Towards describing the strong-coupling regime of the Kardar-Parisi-Zhang (KPZ) equation

1995 J. Phys.: Condens. Matter 7 L605

(http://iopscience.iop.org/0953-8984/7/45/001)

J. Phys.: Condens. Matter 7 (1995) L605–L609. Printed in the UK

LETTER TO THE EDITOR

Towards describing the strong-coupling regime of the Kardar–Parisi–Zhang (KPZ) equation

Semjon Stepanow†
Martin-Luther-Universität Halle-Wittenberg, Fachbereich Physik, Friedemann-Bach-Platz 6, D-06108 Halle/Saale, Germany†

Received 18 September 1995

Abstract. A novel approach which enables one to treat the strong-coupling regime of the KPZ equation is proposed. The essence of the method consists in understanding the reason for the increase of the effective coupling constant under renormalization and incorporating this information into the renormalization group (RG) procedure. By using this generalized RG method I computed the critical exponents at the strong-coupling fixed point of the KPZ equation for space dimensions $2 \leqslant d \leqslant 4$: the roughness exponent $\zeta = (4-d)/4$ and the dynamic exponent $z = (4+d)/4$.

The scale invariance in different topics in physics—such as the physics of phase transitions [1], quantum field theory and the physics of elementary particles [2] and dynamical phenomena far from equilibrium [3]—is usually described by using the renormalization group (RG) method. However, there are examples were the RG methods fails. The most prominent example is probably the KPZ equation introduced and studied by Kardar, Parisi, and Zhang [4]:

$$\frac{\partial h}{\partial t} = \nu_0 \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(x, t) \tag{1}$$

where $h(x, t)$ is a single-valued function which describes the height profile above a basal d-dimensional substrate x in the comoving coordinate system, λ is responsible for the lateral growth, ν_0 is the surface tension, and the noise $\eta(x, t)$ has a Gaussian distribution with $\langle \eta(x, t) \rangle = 0$, and

$$\langle \eta(x, t)\eta(x', t') \rangle = 2D_0\, \delta^d(x - x')\, \delta(t - t'). \tag{2}$$

Equation (1) is now widely accepted as describing growth processes such as the Eden model process and growth by ballistic deposition. The KPZ equation is also related to randomly stirred fluids (Burgers' equation [5]), dissipative transport in the driven-diffusion equation [6], the directed polymer problem in disordered media [7], and the behaviour of flux lines in superconductors [8].

The one-loop renormalization group (RG) flow equations associated with equation (1) can be obtained in a standard manner. The solution of the flow equations is

$$\nu = \nu_0 \left(1 + \frac{1}{\varepsilon} g_0 \beta l^{\varepsilon}\right)^{\gamma_\nu/\beta} \tag{3}$$

$$D = D_0 \left(1 + \frac{1}{\varepsilon} g_0 \beta l^{\varepsilon}\right)^{\gamma_D/\beta} \tag{4}$$

† e-mail: stepanow@physik.uni-halle.d400.de.

$$g = g_0 \Big/ \left(1 + \frac{1}{\varepsilon} g_0 \beta l^{\varepsilon}\right) \tag{5}$$

where $\varepsilon = 2-d$ and $g_0 = K_d D_0 \lambda^2 / \nu_0^3$ is the bare coupling constant. The structure factors γ_χ, γ_D, and β (β is the coefficient in the expansion of the Gell-Mann–Low function in powers of the coupling constant) are computed up to the one-loop order as $\gamma_\chi = (2-d)/(4d)$, $\gamma_D = 1/4$, and $\beta = (3-2d)/(2d)$. In one dimension, the factor β is positive, so the effective coupling constant gl^{ε} has a fixed point, which leads to the scaling behaviour of the effective quantities ν and D. The situation drastically changes in the vicinity of two dimensions. In this case β becomes negative, meaning that the effective coupling constant increases under renormalization and there is no fixed point of the effective coupling constant. However, numerical simulation [10] shows that the interface is rough at $d = 2$. Above two dimensions the interface becomes rough when the coupling g_0 exceeds some critical value g_c. So far, the theory fails to treat this situation. The general opinion is that the strong-coupling fixed point is not obtainable via perturbation theory.

In this letter I will show that the information available from perturbation theory (equations (3)–(5)), combined with an understanding of the reason for the increase of the effective coupling constant under renormalization and incorporating this information into the RG procedure, will enable us to describe the strong-coupling regime of the KPZ equation and in this way compute the critical exponents of the KPZ equation in the strong-coupling regime.

The essential point of the present approach consists in understanding the reason for the increase of the effective coupling constant under renormalization and incorporating this information into the renormalization group procedure. Let us first summarize the main features of the RG method. The method is based on a consecutive integrating out of degrees of freedom along the length scale (shell integration). The integrated degrees of freedom cause renormalizing of the coefficients of the microscopic (bare) model. The RG is expected to be successful if the physics on a macroscopic scale can be understood in terms of the bare equation with renormalized coefficients, which depend on the length scale via power laws. The method definitely fails if this is not the case and physics on the macroscopic scale cannot be described in terms of the bare model i.e. the physics on the macroscopic scale goes beyond the form of the microscopic model. I conclude that, in such cases, the increase of the coupling constant is a 'response' of the system on the shell integration. This conclusion is supported by following two examples. The first one concerns the dynamics of depinning of interfaces and charge density waves (CDW) in disordered media, which was recently successfully treated in [11] (CDW) and [12] (interfaces). The interface will be pinned if the driving force F becomes lower than some threshold value F_c. The microscopic equation (see [12]), however, does not contain the threshold force. Thus, F_c is not present in the bare model, but it is an essential feature of the physics on a macroscopic scale. I interpret this circumstance as being responsible for an increase of the coupling constant under renormalization! The success in solving the depinning problem is connected with the fact that in the depinning problem the renormalization procedure does not reduce to the renormalization of one coupling constant (the strength of the disorder) as is usually the case, but includes the renormalization of the shape of the disorder correlator, which is described by the functional renormalization group [11, 12]. It appears that taking into account the possibility of the appearance of the threshold force F_c in solving the fixed-point equation of the functional renormalization group leads to the fixed point, generating critical behaviour of the interface in the vicinity of the threshold.

The second example concerns the bound-state problem of a quantum mechanical particle in an attractive d-dimensional delta potential, which is equivalent to the problem of a localization of a flux line on a linear defect in d-dimensions and to the absorption of a Gaussian polymer chain on a d-dimensional delta potential. This problem can be studied both exactly, in terms of the well-known Green function of the problem, and by using the dynamical RG, which gives an increase of the effective coupling constant at $d = 2$. Comparing the one-loop RG result with the exact solution tells us (i) that the one-loop result is exact, and (ii) how to use the one-loop result to describe the physics on the macroscopic scale (localization) [13]. It appears that in this case the mechanism for increasing the effective coupling constant under renormalization is due to the fact that for $d > 2$ a finite strength of attraction is necessary for the appearance of a bound state.

Returning to the consideration of the KPZ equation, I assume that the increase of the coupling constant at $d \leqslant 2$ is a signal that for $d > 2$ a finite strength of the coupling constant is necessary in order that the interface will become rough. If so, it is reasonable to incorporate the possibility of generating a threshold of the coupling constant into the RG procedure, and hope that this will result in the appearance of a fixed point—*the strong-coupling fixed point*.

It seems that there is a similarity between the mechanism involved in an increase of the effective coupling constant in the localization problem and that in the KPZ equation. The following arguments give support for this similarity: (i) in both cases there is a finite threshold of the coupling constant for $d > 2$, (ii) there is an analogy between the bound-state problem and the KPZ equation mapped to the directed polymer problem [7]; (iii) there are no higher-order corrections to the Gell-Mann–Low beta function for both the KPZ equation [9] and the localization problem [13].

Let us now start to analyse the problem. The critical dimension $d = 2$ plays a crucial role in the present analysis. From equation (3) it follows that in the vicinity of $d = 2$ only the strength of the thermal noise D renormalizes. In the vicinity of $d = 2$, equation (4) can be written as

$$D = D_0 \left(1 - \frac{1}{\varepsilon} g_0 \beta' l^{\varepsilon}\right)^{-\gamma_D/\beta'} \tag{6}$$

where $\beta' = 1/4$ is the value of the factor β taken at $d = 2$ with the sign negative. The crucial point of the present approach is to identify the length l in equation (6) as

$$l = (\nu_0^{-1}(t^{-1} + t_c^{-1}))^{-1/2} \tag{7}$$

where t is time and t_c is the timescale, which will be defined below. Equation (7) does not contradict perturbation theory on small scales ($t^{-1} \gg t_c^{-1}$) and gives the system the possibility generating a threshold for the coupling constant at $d > 2$ for $t^{-1} \ll t_c^{-1}$. Inserting equation (7) into equation (6) gives

$$D = D_0 \left(\frac{1}{1 - (1/\varepsilon) g_0 \beta' ((\nu_0^{-1}(t^{-1} + t_c^{-1}))^{-\varepsilon/2}}\right)^{\gamma_D/\beta'}. \tag{8}$$

The length t_c is defined by demanding that the denominator in (8) behaves linearly in t^{-1} for small t^{-1}. As a result equation (8) yields for small t^{-1}

$$D = D_0 \left(\frac{2}{g_0 \beta' (\nu_0 t_c)^{(4-d)/2} \nu_0^{-1} t^{-1}}\right)^{\gamma_D/\beta'} \tag{9}$$

with t_c obtained in different dimensions as

$$\nu_0 t_c = \begin{cases} (\varepsilon/(\beta' g_0))^{2/(2-d)} & d < 2 \\ \lambda^2 \exp(2/(\beta' g_0)) & d = 2 \\ [\beta'/((d-2)(g_c^{-1} - g_0^{-1}))]^{2/(d-2)} & d > 2 \end{cases} \tag{10}$$

where $g_c = ((d-2)/\beta')\lambda^{(d-2)}$ is the threshold value of the coupling constant and λ is a microscopic cutoff. It is supposed that in (10) for $d > 2$ the condition $g_0 > g_c$ applies. t_c is the timescale above which the strong-coupling regime is found.

I now assume that equation (9) describes the strong-coupling regime and use this assumption to compute the critical exponents in the strong-coupling regime. Neglecting the prefactor in equation (9) gives

$$D \sim t^{\gamma_D/\beta'}. \tag{11}$$

In contrast to in the weak-coupling regime—where ν at $d = 2$ does not renormalize—in the strong-coupling regime one cannot exclude the possibility that ν may renormalize, so the following scaling behaviour of ν is assumed:

$$\nu \sim l^{\gamma_x/\beta'}. \tag{12}$$

Using the relation $l = (\nu t)^{1/2}$ and equation (12), the relation between t and the spatial scale l becomes $l \sim t^{1/(2-\gamma_x/\beta')}$, which gives the following relation for the dynamic exponent z:

$$z = 2 - \gamma_x/\beta'. \tag{13}$$

Inserting $t \sim l^z$ into equation (11) gives

$$D \sim l^{z\gamma_D/\beta'}. \tag{14}$$

The roughness exponent ζ is derived from the bare expression of the roughness of the surface with D_0 and ν_0 replaced by their renormalized counterparts D and ν:

$$w^2 \simeq l^{2-d} D/\nu \sim l^{2-d+z\gamma_D/\beta'-\gamma_x/\beta'},$$

which gives the roughness as

$$\zeta = \frac{1}{2}(2 - d + z\gamma_D/\beta' - \gamma_x/\beta'). \tag{15}$$

To determine γ_x I use the scaling relation between the exponents, $\zeta + z = 2$, which follows from the invariance of equation (1) to an infinitesimal tilting of the surface $h \to h + \epsilon x$, $x \to x + \lambda\epsilon t$ [4], and after inserting (13) and (15) into the scaling relation I get

$$\frac{\gamma_x}{\beta'} = \frac{2 - d + 2\gamma_D/\beta'}{3 + \gamma_D/\beta'}, \tag{16}$$

which by using $\gamma_D/\beta' = 1$ at $d = 2$ results in

$$\frac{\gamma_x}{\beta'} = \frac{4-d}{4}. \tag{17}$$

On using (17) and $\gamma_D/\beta' = 1$, the critical exponents are obtained from equations (13 and 15) as

$$z = \frac{4+d}{4} \qquad \zeta = \frac{4-d}{4} \qquad \beta = \frac{\zeta}{z} = \frac{4-d}{4+d}. \tag{18}$$

I note that the exponents at $d = 2$ coincide with those conjectured in the original article by Kardar, Parisi, and Zhang. The exponents are larger than those obtained in numerical simulations [10] and those given by analytical formulae conjectured by Wolf and Kertész [14] and Kim and Kosterlitz [15], which were obtained by using the results of numerical simulations. The discrepancy may be due to the fact that simulations were not performed in the scaling regime. The above time scale t_c which the strong-coupling regime develops, depends on g_0 in a nontrivial way (see equation (10)), so t_c can become extremely large, which would make it difficult to achieve the scaling regime in computer simulations. It is interesting to note that $d = 4$ is the upper critical dimension for the strong-coupling regime. The same conclusion was drawn in the previous studies [9, 16]–[17]. Due to the fact that there are no singularities beyond the one-loop level, I expect that the critical exponents (18) will be exact for $d \geqslant 2$. It is evident from the above consideration that the present method, being appropriate for the treatment of the strong-coupling regime, is expected to describe the exponents at $d = 1$ only approximately.

Let us now discuss the peculiarities of the strong-coupling problem associated with the KPZ equation. It was argued above that there is some similarity between the strong-coupling regime of the KPZ equation and the bound-state problem of quantum mechanics but the latter is essentially a one-particle problem. The main difficulty in treating the KPZ equation lies in recognizing the feature of 'the one-particle problem' in the formalism of infinite degrees of freedom associated with the KPZ equation and in recognizing how 'the one-particle problem' manifests itself in the RG procedure.

To conclude, I have proposed a novel approach enabling one to treat the strong-coupling regime of the KPZ equation. Although the present mechanism of treating the strong-coupling regime is definitely not generic for all cases were the effective coupling constant increases under renormalization (compare the KPZ case and interface depinning), I expect that the basic idea of the present work *is* generic and will be useful in treating other problems in different topics in physics where the strong coupling is relevant.

I acknowledge partial support from the Deutsche Forschungsgemeinschaft (SFB 262, Mainz) where part of this work was performed.

References

[1] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Oxford University Press)
[2] Bogoliubov N N and Shirkov D V 1976 *Introduction to the Theory of Quantized Fields* (Moscow: Nauka)
[3] Krug J and Spohn H 1990 *Solids far from Equilibrium* ed C Godrèche (New York: Cambridge University Press)
[4] Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56** 889
[5] Forster D, Nelson D R and Stephen M J *Phys. Rev.* A **16** 732
[6] van Beijeren H, Kutner R and Spohn H 1985 *Phys. Rev. Lett.* **54** 2036
Jansen H K and Schmittmann B 1986 *Z. Phys.* B **63** 517
[7] Kardar M and Zhang Y C 1987 *Phys. Rev. Lett.* **58** 2087
[8] Hwa T 1992 *Phys. Rev. Lett.* **69** 1552
[9] 1994 Lässig M *Preprint*
[10] Forrest B M and Tang L H 1990 *Phys. Rev. Lett.* **64** 1405
Ala-Nissala T, Hjet T and Kosterlitz J M 1992 *Europhys. Lett.* **19** 1
[11] Narayan O and Fisher D S 1992 *Phys. Rev. Lett.* **68** 3615
[12] Nattermann T, Stepanow S, Tang L H and Leschhorn H 1992 *J. Physique II* **2** 1483
Narayan O and Fisher D S 1993 *Phys. Rev.* B **48** 7030
[13] Stepanow S unpublished
[14] Wolf D E and Kertész J 1987 *Europhys. Lett.* **4** 651
[15] Kim J M and Kosterlitz J M 1989 *Phys. Rev. Lett.* **62** 2289
[16] Frey E and Täuber U C 1994 *Phys. Rev.* E **50** 1024
[17] Moore M A, Blum T, Doherty J P, Marsili M, Bouchaud J-P and Claudin P 1995 *Phys. Rev. Lett.* **74** 4257